Introduction to Smooth Manifolds – Chapter 4

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1 Exercises

Exercise 4.3

(a)

Proof. Let $M = M_1 \times \cdots \times M_k$ and $\dim(M_i) = n_i$. To prove that π_i is a submersion, we need to show that $\text{rank}(d\pi_i)_p = n_i$ for all $p \in M$. We have that $\pi_i : M_1 \times \cdots \times M_k \to M_i$ is the projection function defined by

$$\pi_i(p_1, \ldots, p_k) = p_i$$

It was previously proven that π_i is smooth. Furthermore, by definition, we can write

$$\iota \circ \pi_i = \text{Id}_M\big|_{M_i}$$

So we have

$$d\iota \circ d\pi_i = d(\text{Id}_M\big|_{M_i})$$

By Prop. 3.6, $d(\text{Id}_M\big|_{M_i})_p = \text{Id}_{T_{p_i}M_i}$, so $\text{rank}(d(\text{Id}_M\big|_{M_i})_p) = n_i$. Additionally,

M_i is an open submanifold of $\iota(M_i)$. Thus, by Prop. 3.9, $d\iota_{p_i}$ is an isomorphism, $d\iota_{p_i} : T_{p_i}M_i \to T_{\iota(p_i)}M$, so $\text{rank}(d\iota_{p_i}) = n_i$. By the properties of a composition of linear maps, $\text{rank}((d\pi_i)_p) = n_i$. Since p was arbitrary, this holds for all $p \in M$, so π_i is a smooth submersion. □

Exercise 4.4

Proof. This is a natural consequence of composition of linear maps. Let $f : M \to N$ and $g : N \to P$ be smooth submersions. Consider, $d(g \circ f)_p = dg_{f(p)} \circ df_p$ for any $p \in M$. Then $\text{rank}(df_p) = \dim(N)$ and $\text{rank}(dg_{f(p)}) = \dim(P)$, so we can view $dg_{f(p)} \circ df_p$ as a multiplication of Jacobian matrices (with respect to some choice of coordinate charts) of full rank, and thus $\text{rank}(d(g \circ f)_p) = \dim(P)$, so $g \circ f$ is a smooth submersion.

The same goes for immersions. If f and g are immersions, then $\dim(\ker(df_p)) =$

0 and $\dim(\ker(dg_{f(p)})) = 0$, so by composition of linear maps, $\dim(\ker(d(g \circ f)_p)) = 0$, so $g \circ f$ is a smooth immersion. □

Now, we need to devise a counterexample to show that a composition of maps of constant rank need not have constant rank. Define $f : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (t, t^2)$ and $g : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto y$. Then f and g both have constant rank 1, but $g \circ f$ has rank 1 at $t \neq 0$ but rank 0 at $t = 0$. Credit for this part goes to: Georges Elencwajg

Exercise 4.16

Show that every composition of smooth embeddings is a smooth embedding.

Proof. A smooth embedding between manifolds M and N is defined as a smooth immersion $F : M \to N$ that is a homeomorphism on its image $F(M) \subseteq N$ in the subspace topology. Let $F : M \to N$ and $G : N \to P$ be smooth embeddings.

We know already that the composition of smooth maps is smooth. Then let $v \in T_{G\circ F(p)}P$ for $p \in M$. Since G is an immersion, there exists a uniqe $q \in T_{F(p)}N$ such that $dG_{F(p)}(q) = v$. Similarly, since F is an immersion, there exists a unique $u \in T_pM$ such that $dF_p(u) = q$. Thus $d(G \circ F)_p(u) = dG_{F(p)} \circ dF_p = v$ for a unique $u \in T_pM$ which shows that $G \circ F$ is a smooth immersion.

Now, consider $G \circ F : M \to P$. It is a continuous mapping because the composition of continuous maps is continuous. It is bijective, because the composition of bijections is a bijection. And its inverse, $F^{-1} \circ G^{-1}$ is continuous, because it is a composition of continuous inverses. Thus, it is a homeomorphism onto its image and therefore a smooth embedding. □

Exercise 4.24

Give an example of a smooth embedding that is neither an open map nor a closed map.

Proof. Define $f : (0, 1) \to \mathbb{R}^2$ by $f(x) = (x, 0)$. $(0, 1)$ considered as a subset of $(0, 1)$ is both open and closed, but $f((0, 1))$ is neither open nor closed as a subset of $\mathbb{R}^2$. However, it is open and closed as a subset of its image, $(0, 1) \times \{0\}$, so it is a smooth embedding. □

2 Problems

Problem 4.2

Proof. Let $p \in M$ such that dF_p is nonsingular. We can choose charts (U, φ) for $p \in M$ and (V, ψ) for $F(p) \in N$. Define $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$, $\hat{V} = \psi(V) \subseteq \mathbb{H}^n$, and $\hat{F} = \psi \circ F \circ \varphi^{-1}$. $\hat{U}$ and $\hat{V}$ are open when considered as subsets of $\mathbb{R}^n$

and $\mathbb{H}^n$, respectively. However, to apply the Inverse Function Theorem in $\mathbb{R}^n$, we need that $\hat{V}$ be considered as an open subset of $\mathbb{R}^n$. We can canonically identify $T_{\hat{F}(x)}\hat{V}$ with $T_{\hat{F}(x)}\mathbb{H}^n$ where $p = \varphi^{-1}(x)$. However, we must apply the inclusion function $\iota : \mathbb{H}^n \hookrightarrow \mathbb{R}^n$ to consider $\hat{V}$ as a subset of $\mathbb{R}^n$. By Lemma 3.11, $d\iota_{\hat{F}(x)} : T_{\hat{F}(x)}\mathbb{H}^n \to T_{\hat{F}x}\mathbb{R}^n$ is an isomorphism, so $d\iota_{\hat{F}(x)}$ is also nonsingular. Thus, since $d\iota_{\hat{F}(x)} \circ d\hat{F}_x$ is nonsingular, we can apply the IFT to $\iota \circ \hat{F}$ to show that it is a diffeomorphism from $\hat{U}$ to some neighborhood of $\iota(\hat{F}(x))$, now both considered as subsets of $\mathbb{R}^n$. However, since $\hat{F}(x) \in \partial\mathbb{H}^n$, there does not exist an open neighborhood V' of $\hat{F}(x)$ such that $\iota(V')$ is a neighborhood of $\iota(\hat{F}(x)) \subseteq \mathbb{R}^n$, which is a contradiction. Therefore, we must have that $p \in \mathrm{Int}(N)$. $\square$